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Logarithmic moments of characteristic polynomials of random matrices

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Dedicated for his birthday to our very distinguished colleague and dear friend Joel Lebowitz, the scientist and the untamable militant for human rights

Abstract

In a recent article we have discussed the connections between averages of powers of Riemann's ζ -function on the critical line, and averages of characteristic polynomials of random matrices. The result for random matrices was shown to be universal, i.e. independent of the specific probability distribution, and the results were derived for arbitrary moments. This allows one to extend the previous results to logarithmic moments, for which we derive the explicit universal expressions in random matrix theory. We then compare these results to various results and conjectures for ζ -functions, and the correspondence is again striking.

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1 Correlation functions of characteristic polynomials

We first briefly review the result of a previous paper [1], in which we have investigated the average of a product of characteristic polynomials of a random matrix. Let X be an $M \times M$ random Hermitian matrix. The correlation function of $2K$ distinct characteristic polynomials is defined as

$$F_{2K}(\lambda_1, \dots, \lambda_{2K}) = \langle \prod_{\alpha=1}^{2K} \det(\lambda_\alpha - X) \rangle. \quad (1)$$

The average is taken with the normalized probability distribution

$$P(X) = \frac{1}{Z} \exp -N \text{Tr} V(X), \quad (2)$$

where $V(X)$ is a polynomial in X . The simplest case consists of a Gaussian distribution, and the result is easily worked out by the orthogonal polynomial method,

$$P(X) = \frac{1}{Z_M} \exp -\frac{N}{2} \text{Tr} X^2, \quad (3)$$

with

$$M = N - K. \quad (4)$$

Defining the middle-point

$$\lambda = \frac{1}{2K} \sum_{\alpha=1}^{2K} \lambda_\alpha, \quad (5)$$

and the density of eigenvalues at this point

$$\rho(\lambda) = \frac{1}{2\pi} \sqrt{4 - \lambda^2}, \quad (6)$$

we introduced the scaling variables

$$x_a = 2\pi N \rho(\lambda) (\lambda_a - \lambda), \text{ with } \sum_{a=1}^{2K} x_a = 0, \quad (7)$$

and consider the large N limit, finite x_a 's limit. In this limit we have shown [1] that,

$$\begin{aligned} & \exp -\left(\frac{N}{2} \sum_{l=1}^{2K} V(\lambda_l)\right) F_{2K}(\lambda_1, \dots, \lambda_{2K}) = \\ & (2\pi N \rho(\lambda))^{K^2} \frac{\exp(-NK)}{K!} \oint \prod_1^K \frac{du_\alpha}{2\pi} \exp -i\left(\sum_{\alpha=1}^K u_\alpha\right) \frac{\Delta^2(u_1, \dots, u_K)}{\prod_{\alpha=1}^K \prod_{l=1}^{2K} (u_\alpha - x_l)}. \end{aligned} \quad (8)$$

in which the contours enclose all the x_l 's. It has also been shown that the result is universal in this scaling limit, i.e. independent of the specific polynomial V which defines the probability distribution. For $K = 1$, we have

$$\exp\left\{-\frac{N}{2}(V(\lambda_1) + V(\lambda_2))\right\}F_2(\lambda_1, \lambda_2) = e^{-N}(2\pi N\rho(\lambda))\frac{\sin x}{x} \quad (9)$$

with $x = \pi N\rho(\lambda)(\lambda_1 - \lambda_2)$, the familiar sine-kernel.

When all the λ_i 's are equal, we obtain the $2K$ -th moment of the characteristic polynomial :

$$\exp(-(NKV(\lambda))F_{2K}(\lambda, \dots, \lambda) = (2\pi N\rho(\lambda))^{K^2} \frac{\exp(-NK)}{K!} \oint \prod_1^K \frac{du_\alpha}{2\pi} \exp -i\left(\sum_{\alpha=1}^K u_\alpha\right) \frac{\Delta^2(u_1, \dots, u_K)}{\prod_{\alpha=1}^K u_\alpha^{2K}}. \quad (10)$$

This contour integration reduces to a simple determinant and one finds,

$$\exp(-(NKV(\lambda))F_{2K}(\lambda, \dots, \lambda) = (2\pi N\rho(\lambda))^{K^2} e^{-NK} \prod_0^{K-1} \frac{l!}{(K+l)!}. \quad (11)$$

Let us denote the last factor by

$$\gamma_K = \prod_0^{K-1} \frac{l!}{(K+l)!}. \quad (12)$$

In our previous paper [1], we have compared (11) with the average of the $2K$ -th moment of the ζ -function [2, 3], for which it has been conjectured that

$$\frac{1}{T} \int_0^T dt |\zeta(\frac{1}{2} + it)|^{2K} \simeq \gamma_K a_K (\log T)^{K^2} \quad (13)$$

where a_K is a number theoretic coefficient given by the product of the prime p ,

$$a_K = \prod_p [(1 - \frac{1}{p})^{K^2} \sum_{m=0}^{\infty} (\frac{K(K+1) \cdots (K+m-1)}{m!})^2 p^{-m}], \quad (14)$$

and γ_K is the same as in (12). Since we have with (11) an expression valid for all K 's, we shall extend this comparison between the moments of characteristic polynomials and that of zeta-functions to non-integer K , as will be explained in the next section.

2 Non-integer power moment

Since the result (11) is valid for any K we may now consider the analytic continuation to non-integer K . This requires to continue the coefficient γ_K in (12) to non-integer values. This will be needed for obtaining the logarithmic moment of the characteristic polynomials. The non-integer power moments are also interesting by themselves, since there exists equivalent studies of fractional power moments of the Riemann ζ -function [4].

The factor $\gamma_K = \prod l!/(K+l)!$ may be expressed through an integral representation,

$$\log \gamma_K = - \int_0^\infty dt \frac{e^{-t}}{t} \left[K^2 - \frac{(1 - e^{-Kt})^2}{(1 - e^{-t})^2} \right] \quad (15)$$

which is easily checked by expanding the integrand in powers of e^{-t} . It is more conveniently handled if we take it as

$$\log \gamma_K = \lim_{\alpha \rightarrow 1} - \int_0^\infty dt \frac{e^{-t}}{t^\alpha} \left[K^2 - \frac{(1 - e^{-Kt})^2}{(1 - e^{-t})^2} \right], \quad (16)$$

since it may then be splitted into two parts for $\alpha < 1$.

Expanding the integrand in the power of e^{-t} , and integrating over t , we obtain

$$\begin{aligned} \log \gamma_K &= -\Gamma(1-\alpha) \left[K^2 - \sum_{n=0}^{\infty} (n+1)^\alpha + 2 \sum_{n=0}^{\infty} (n+K+1)^{\alpha-1} (n+1) \right. \\ &\quad \left. - \sum_{n=0}^{\infty} (n+2K+1)^{\alpha-1} (n+1) \right] \\ &= -\Gamma(1-\alpha) \left[K^2 - \zeta(-\alpha) + 2\zeta(-\alpha, 2K+1) - 2K\zeta(1-\alpha, K+1) \right. \\ &\quad \left. - \zeta(-\alpha, 2K+1) + 2K\zeta(1-\alpha, 2K+1) \right] \end{aligned} \quad (17)$$

in which the limit $\alpha \rightarrow 1$ is meant. The generalized zeta-function $\zeta(z, a)$ is given by $\sum_{n=0}^{\infty} (a+n)^{-z}$, and $\zeta(0, a) = \frac{1}{2} - a$; it has the expansion

$$\begin{aligned} \zeta(z, a) &= \frac{2\Gamma(1-z)}{(2\pi)^{1-z}} \left[\sin\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\cos(2n\pi a)}{n^{1-z}} \right. \\ &\quad \left. + \cos\left(\frac{\pi z}{2}\right) \sum_{n=1}^{\infty} \frac{\sin(2n\pi a)}{n^{1-z}} \right]. \end{aligned} \quad (18)$$

It is then easy to obtain various results for non-integer K . For instance the $K = 0$ limit is obtained by expanding in powers of K . The term of order K^2 is

$$\begin{aligned}\log \gamma_K &\simeq -K^2[\Gamma(1-\alpha) - \Gamma(3-\alpha) \sum_{n=0}^{\infty} \frac{1}{(n+1)^{2-\alpha}}] \\ &= -K^2[\Gamma(1-\alpha) - \Gamma(3-\alpha)\zeta(2-\alpha)] \\ &= K^2(1+c) + O(K^3)\end{aligned}\tag{19}$$

where c is Euler's constant, $c = 0.5772\dots$

It is also interesting to evaluate γ_K for $K = \frac{1}{2}$, since it is needed for computing the first moment of the characteristic polynomial or of the Riemann ζ -function. In this case, we have

$$\begin{aligned}\log \gamma_{\frac{1}{2}} &= \lim_{\alpha \rightarrow 1} - \int_0^{\infty} \left(\frac{1}{4} - \frac{1}{(1 + e^{-t/2})^2} \right) \frac{e^{-t}}{t^{\alpha}} dt \\ &= \lim_{\alpha \rightarrow 1} -\frac{1}{4}\Gamma(1-\alpha) - 2^{1-\alpha} \sum_{n=0}^{\infty} (-1)^n [(n+2)^{\alpha} - (n+2)^{\alpha-1}] \Gamma(1-\alpha) \\ &= \lim_{\alpha \rightarrow 1} \left[-\frac{1}{4} + 2^{1-\alpha} ((2^{1+\alpha} - 1)\zeta(-\alpha) - (2^{\alpha} - 1)\zeta(1-\alpha)\Gamma(1-\alpha)) \right] \\ &= \frac{1}{12} \log 2 + \frac{1}{2} \log \pi + 3\zeta'(-1)\end{aligned}\tag{20}$$

since $\zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$, and $\sum_{n=0}^{\infty} (-1)^n (n+2)^{-s} = 1 + (2^{1-s} - 1)\zeta(s)$. This leads to $\gamma_{\frac{1}{2}}$ is 1.1432.... In the literature on Riemann ζ -functions concerning the moments (13), bounds have been conjectured for $0 \leq K \leq 1$ [4, 5], and they amount for the equivalent of γ_K to

$$\frac{1}{\Gamma(K^2 + 1)} \leq \gamma_K \leq \frac{2}{\Gamma(K^2 + 2)(2 - K)}.\tag{21}$$

We find that our result of (15) indeed satisfies this bound for $0 \leq K \leq 1$. For instance, the bounds require $1.1033 \leq \gamma_{\frac{1}{2}} \leq 1.1768$, and we have found $\gamma_{\frac{1}{2}} = 1.1432$. It is easy to verify this bound by expanding around $K = 1$ and $K = 0$, and it does support the conjecture (21).

3 Moment of the logarithm

We now consider the case for which all the λ_α 's are equal, and expand the $2K$ -th moment in powers of K :

$$\begin{aligned}
I &= F_{2K}(\lambda, \dots, \lambda) e^{-NKV(\lambda) + NK} \\
&= \langle [\det(\lambda - X) e^{-\frac{N}{2}V(\lambda) + \frac{N}{2}}]^{2K} \rangle \\
&= \sum_{p=0}^{\infty} \frac{(2K)^p}{p!} \langle [\log |\det(\lambda - X)| - \frac{N}{2}V(\lambda) + \frac{N}{2}]^p \rangle.
\end{aligned} \tag{22}$$

From (11), we know the same I as

$$I = \sum_{p'=0}^{\infty} \frac{K^{2p'}}{p'!} [\log(2\pi N\rho(\lambda))]^{p'} \tag{23}$$

provided we neglect, for the moment, the factor γ_K and set it equal to one. We have thus made an analytic continuation from integer K to a real variable K , and we have expanded in K . We have seen in the previous section that, for K small,

$$\gamma_K = 1 + K^2(1 + c) + O(K^3) \tag{24}$$

where c is Euler constant. This correction of order K^2 gives only a subleading term in (23) compared to $\log(2\pi N\rho(\lambda))$, and thus we were justified to neglect it.

Comparing (22) and (23), we find

$$\langle [\log |\det(\lambda - X)| - \frac{N}{2}V(\lambda) + \frac{N}{2}]^{2m} \rangle \simeq \frac{1}{2^{2m}} \frac{2m!}{m!} [\log(2\pi N\rho(\lambda))]^m \tag{25}$$

Since there is the expansion (23) contains only even powers of K , we have also,

$$\langle [\log |\det(\lambda - X)| - \frac{N}{2}V(\lambda) + \frac{N}{2}]^{2m+1} \rangle = 0 \tag{26}$$

It is instructive to verify this for $m = 0$; taking a derivative of (26) with respect to λ one should verify that

$$\begin{aligned}
\langle \text{Tr}(\frac{1}{\lambda - X}) \rangle &= \int da \frac{\rho(a)}{\lambda - a} \\
&= \frac{N}{2} V'(\lambda)
\end{aligned} \tag{27}$$

which is nothing but the saddle point equation of the large N limit which determines $\rho(\lambda)$ [6]. The solution of this Riemann-Hilbert problem is expressed through $G(z) = \frac{1}{N} < \text{Tr}(\frac{1}{z-X}) > = (\frac{1}{2}V'(z) - P(z)\sqrt{Q(z)})$, in which P and Q are polynomials fixed by the requirement that G falls like $1/z$ at infinity. Then $\text{Re } G(x + i\epsilon) = V'(x)$, $\text{Im } G(x + i\epsilon) = -\pi\rho(x)$ in the large N limit. The r.h.s. of the above equation is indeed $N\text{Re}G(\lambda) = \frac{N}{2}V'(\lambda)$.

Our argument may be applied to the moment of the ζ -function as well. The average of $2K$ moment of ζ function has been conjectured [2, 3] as

$$\frac{1}{T} \int_0^T dt |\zeta(\frac{1}{2} + it)|^{2K} \simeq \gamma_K a_K (\log T)^{K^2} \quad (28)$$

Expanding the above equation in powers of K , we obtain along the same lines

$$\frac{1}{T} \int_0^T [\log |\zeta(\frac{1}{2} + it)|]^{2m} \simeq \frac{1}{2^{2m}} \frac{2m!}{m!} (\log \log T)^m \quad (29)$$

provided a_K goes to one in the limit $K \rightarrow 0$. Selberg [7] has derived

$$\frac{1}{T} \int_0^T [\text{Im} \log(\zeta(\frac{1}{2} + it))]^{2m} \sim \frac{1}{2^{2m}} \frac{2m!}{m!} (\log \log T)^m \quad (30)$$

where $\text{Im} \log \zeta(x) = \arg \zeta(x)$.

Our result (25) may be compared to (29) or (30), in which the density of state $2\pi N\rho(\lambda)$ is replaced by $\log T$. The characteristic polynomial $\det(\lambda - X)$ has zeros on the real axis in the complex plane of λ . Hence this function corresponds to Riemann's $\xi(\frac{1}{2} + i\lambda) = A(\lambda)Z(\lambda)$, which has zeros on the real λ line, where

$$A(\lambda) = \pi^{-\frac{1}{4}} e^{\text{Re} \log \Gamma(\frac{1}{4} + \frac{i\lambda}{2})} (-\frac{\lambda^2}{2} - \frac{1}{4}), \quad (31)$$

$$Z(\lambda) = e^{i\theta} \zeta(\frac{1}{2} + i\lambda). \quad (32)$$

The analogy between $|\det(\lambda - X)|$ and $|\xi(\frac{1}{2} + i\lambda)| = |A(\lambda)| |\zeta(\frac{1}{2} + i\lambda)|$ leads to $\log |\zeta(\frac{1}{2} + i\lambda)| \sim \log |\det(\lambda - x)| - \log |A(\lambda)|$, and it may thus correspond to $\log |\det(\lambda - X)| - \frac{NV}{2} + \frac{N}{2}$.

From those log-moment results, we find that the distribution function is a normal Gaussian distribution. For the ζ function, or more generally for the L functions, it

is known that $\log |L(\frac{1}{2} + it)|$ is distributed like a random variable, when t is large, with a Gaussian density [7, 8, 9]. The coefficient of (25), $2m!/(2^{2m}m!)$ is equal to $\frac{1}{\sqrt{\pi}}\Gamma(m + \frac{1}{2})$, which is identical to the coefficient of the following Gaussian integral,

$$\int_{-\infty}^{\infty} x^{2m} e^{-ax^2} dx = \frac{1}{\sqrt{\pi}} \Gamma(m + \frac{1}{2}) a^{-m-\frac{1}{2}}. \quad (33)$$

Hence we find that our moment (25) does follow a normal distribution.

We have considered up to now the case of all the λ_i 's equal ; in the following we shall consider two different λ_i 's, and it will also appear that a normal distribution holds for the logarithmic moments.

4 Moments at two different points

The formula for F_{2K} in (8) provides also the correlation for two different values of the "energies", λ_1 and λ_2 , if we set $\lambda_a = \lambda_1 = \dots = \lambda_K$ and $\lambda_b = \lambda_{K+1} = \dots = \lambda_{2K}$. For instance, when $K = 2$, ($l_1 = 2, l_2 = 2$, and $l_1 + l_2 = 2K$), we have $x_1 = x_2 = -x_3 = -x_4 = x$. Then, the contour integral in (8) becomes

$$\oint \frac{du_1 du_2}{(2\pi)^2} \frac{e^{-i(u_1+u_2)}(u_1-u_2)^2}{(u_1-x)^2(u_1+x)^2(u_2-x)^2(u_2+x)^2} = \frac{1}{2x^2} \left(1 - \frac{\sin^2 x}{x^2}\right) \quad (34)$$

where $x = \pi N \rho(\lambda)(\lambda_1 - \lambda_2)$, with $\lambda = \frac{1}{2}(\lambda_1 + \lambda_2)$. Note that the contour integral formula of (8) has been derived in Dyson's short distance limit [10, 11]. We consider, within this Dyson limit, the large x case : $x \gg 1$. The leading term for large x , is easily obtained from (8). The subleading terms have an oscillatory behavior, but we limit ourselves for simplicity to the leading term . Then the leading behavior becomes

$$\langle [\det(\lambda_1 - X)]^{2l_1} [\det(\lambda_2 - X)]^{2l_2} \rangle e^{-Nl_1 V(\lambda_1) + l_1 N - Nl_2 V(\lambda_2) + l_2 N} \simeq \frac{1}{x^{\frac{1}{2}(l_1+l_2)^2}} (2\pi N \rho)^{(l_1+l_2)^2} \quad (35)$$

Expanding in powers of l_1 and l_2 , we obtain from the l.h.s. of (35),

$$\begin{aligned} I &= \sum_{p_1, p_2=0}^{\infty} \frac{(2l_1)^{p_1}}{p_1!} \frac{(2l_2)^{p_2}}{p_2!} \langle [\log |\det(\lambda_1 - X)| - \frac{N}{2} V(\lambda_1) + \frac{N}{2}]^{p_1} [\log |\det(\lambda_2 - X)| \\ &\quad - \frac{N}{2} V(\lambda_2) + \frac{N}{2}]^{p_2} \rangle \end{aligned} \quad (36)$$

For the r.h.s. of (35), we have

$$\sum_{p=0}^{\infty} \left(-\frac{1}{2}\right)^p \frac{(l_1 + l_2)^{2p}}{p!} (\log x)^p = \sum_{p=0}^{\infty} \sum_{p_1, p_2, 2p=p_1+p_2} \frac{(2p)!}{p!} \frac{l_1^{p_1}}{p_1!} \frac{l_2^{p_2}}{p_2!} \left[\log\left(\frac{2\pi N\rho}{\sqrt{2x}}\right)\right]^p \quad (37)$$

Hence, we obtain when $p_1 + p_2$ is even integer,

$$\begin{aligned} & \langle [\log |\det(\lambda_1 - X)| - \frac{N}{2}V(\lambda_1) + \frac{N}{2}]^{p_1} [\log |\det(\lambda_2 - X)| - \frac{N}{2}V(\lambda_2) + \frac{N}{2}]^{p_2} \rangle \\ &= \left(\frac{1}{2}\right)^{2p} \frac{(2p)!}{p!} \left[\log\left(\frac{2\pi N\rho}{\sqrt{2x}}\right)\right]^p \end{aligned} \quad (38)$$

where $p = (p_1 + p_2)/2$. When $p_1 + p_2$ is an odd integer, the correlation vanishes since there is no corresponding term in the r.h.s.. The above result has been derived for large x . In this limit, we have obtained,

$$\begin{aligned} & \langle [\log |\det(\lambda_1 - X)| - \frac{N}{2}V(\lambda_1) - \log |\det(\lambda_2 - X)| + \frac{N}{2}V(\lambda_2)]^{2p} \rangle \\ &\simeq \langle [\log |\det(\lambda_1 - X)| - \log |\det(\lambda_2 - X)|]^{2p} \rangle \\ &\simeq 2\left(\frac{1}{2}\right)^{2p} \frac{(2p)!}{p!} (\log[2\pi N\rho])^p - 2\left(\frac{1}{2}\right)^{2p} \frac{(2p)!}{p!} \left(\log\left[\frac{2\pi N\rho}{\sqrt{2x}}\right]\right)^p \end{aligned} \quad (39)$$

where we have expanded the binomial forms, and used (38). The difference $V(\lambda_1) - V(\lambda_2)$ gives a subleading term, and it has been neglected.

In the simple case, $p = 1$ for (39), we have a cross-term with two logarithms. By taking the derivatives of this cross term, we obtain

$$\partial_{\lambda_1} \partial_{\lambda_2} \langle \log |\det(\lambda_1 - X)| \log |\det(\lambda_2 - X)| \rangle = \langle \text{Tr} \frac{1}{\lambda_1 - X} \text{Tr} \frac{1}{\lambda_2 - X} \rangle \quad (40)$$

which is two point Green function. We have

$$\langle \text{Tr} \frac{1}{z_1 - X} \text{Tr} \frac{1}{z_2 - X} \rangle = N^2 G_{2c}(z_1, z_2) + N^2 G(z_1) G(z_2) \quad (41)$$

where the connected two-point Green function $G_{2c}(z_1, z_2)$ has been found in [12, 13].

There it has been shown that

$$\begin{aligned} N^2 G_{2c}(z_1, z_2) &= -\partial_{z_1} \partial_{z_2} \log[1 - G(z_1)G(z_2)] \\ &= \partial_{z_1} \partial_{z_2} \log\left[\frac{u(z_1) - u(z_2)}{z_1 - z_2}\right] \\ &= \frac{1}{2(z_1 - z_2)^2} \left(\frac{z_1 z_2 - 4}{[(z_1^2 - 4)(z_2^2 - 4)]^{1/2}} - 1 \right) \end{aligned} \quad (42)$$

where $u(z) = \frac{1}{2}[z + \sqrt{z^2 - 4}]$. So indeed the result (39) for $p = 1$ is consistent with the previously known results (42) in the large N limit. Note that the result (42) has been derived by taking the large N limit first ; hence it is a smoothed correlation function, which neglects all the oscillatory terms. (The connected two-point correlation function is $\rho_{2c}(\lambda_1, \lambda_2) = -\frac{1}{4\pi^2}[G_{2c}(\lambda_1 + i\epsilon, \lambda_2 + i\epsilon) + G_{2c}(\lambda_1 - i\epsilon, \lambda_2 - i\epsilon) - G_{2c}(\lambda_1 + i\epsilon, \lambda_2 - i\epsilon) - G_{2c}(\lambda_1 - i\epsilon, \lambda_2 + i\epsilon)]$, and it becomes $-1/2\pi^2 N^2(\lambda_1 - \lambda_2)^2$ for λ_1 close to λ_2 . This result is obtained by smoothing the oscillatory part, while the exact result is $\rho_{2c}(\lambda_1, \lambda_2) \simeq -\sin^2 x/\pi^2 N^2(\lambda_1 - \lambda_2)$; by taking the large- N limit first, the $\sin^2 x$ is replaced by $1/2$.)

We noticed that a similar formula exists for the Riemann ζ -function, although it deals with the imaginary part of the logarithm of the ζ -function, from which a study of the variance of the number of zeros has been discussed in the literature [14, 15].

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